4. FIRST STEPS IN THE THEORY

§4.1. A Catalogue of All Groups: Impossible Dream

The fundamental problem of group theory is to systematically explore the landscape and to chart what lies out there. We'd like to have a catalogue of all groups, or perhaps just all finite groups: a catalogue that would provide one specimen of each group so that we'd be able to say that every group, or every finite group, is isomorphic to exactly one of the specimens in our catalogue. In particular we'd know exactly how many groups there are of any given order.

This is an impossible dream, but not simply because there are infinitely many finite groups. After all there are infinitely many finite-dimensional vector spaces over a given field yet we can describe them all in the single statement that every finite-dimensional vector space over a field F is isomorphic to the space of *n*-tuples (x_1, x_2, ... , x_n) over F for some *n*. This is just a corollary of the theorem that every finite-dimensional vector space has a basis. In other words there's exactly one vector space over F (up to isomorphism) for each dimension.

Structure theorems exist in various parts of mathematics. We know all the finite fields. In topology the compact surfaces are all known. Within group theory itself we have a number of classification theorems. For example we know all the finite abelian groups, via the Fundamental Theorem of Abelian Groups that we'll prove in a later chapter.

The most celebrated example is the classification of all finite 'simple' groups. Don't worry now what a simple group is. In a certain sense they're the building blocks of all finite groups but the term 'simple' is deceptive.. They're not the most elementary of groups but their simplicity refers to the fact that they can't be pulled apart, in a certain way, into smaller groups. They're the atoms of the universe of finite groups.

Now this celebrated theorem is the culmination of virtually a century's work by many thousands of group theorists and its proof is scattered over numerous research journals in the literature. It has never been assembled in one place because it's estimated that it would occupy about 20,000 pages and because of this it has even made it to the Guinness Book of Records as the theorem with the longest proof!

The goal of this chapter is to develop enough of the theory to enable us to prove a couple of classification theorems; nothing as ambitious as the one we were just talking about – just a couple of nice, gentle theorems. Together they describe, for each prime p, the groups of order p and of order $2p$. There aren't very many such groups for a given prime. In fact there's only one group of order p and just two of order $2p$. So contrary to what you might expect the number of different groups doesn't grow steadily with the group order. The groups of order 32 have been catalogued and there are over 50 of them. But there's only one group of order 31 because 31 is prime and there are just two groups of order 34.

§4.2. Additive and Multiplicative Notation

Until now we've written the combination of a and b in an abstract group as $a * b$. But, unless there's danger of confusion, it's more usual to write the binary operation as if it was addition or multiplication.

If we're proving theorems about abstract groups in general, where the group could be non-abelian, we use **multiplicative notation**. If the groups we're considering are all abelian we normally use **additive notation**. The reason for this distinction is that we're used to multiplication being non-commutative (for example matrices and permutations), but in all the systems we've ever encountered, addition is commutative. It's very dangerous therefore to use additive notation in a non-commutative group.

GENERAL NOTATION	**MULTIPLICATIVE NOTATION**	**ADDITIVE NOTATION**
$a * b$	ab	$a + b$
$a * a * \ldots * a$	a^n	na
e	1	0
inverse of a	a^{-1}	$-a$

Just remember that even though we use multiplicative or additive notation the operation might be quite different to ordinary multiplication or addition of numbers.

§4.3. Basic Properties of Groups

Having motivated you in terms of group theory, and giving a huge number of examples of many different types I now intend to make a fresh beginning. Indeed this is where most books on group theory begin. I will restate the definition and slowly build up the theory.

So, recall the definition of a **group** as a set with one associative binary operation for which there is an identity and where each element has an inverse. Notice that there is nothing in this definition that precludes a group from having multiple identities or an element having more than one inverse. But in fact a group can only have one identity, and inverses are unique. Because of this it makes sense to adopt a symbol for *the* identity (1 in multiplicative notation and 0 in additive notation).

Moreover the notation a^{-1} or $-a$ would be misleading if there were multiple inverses.

Theorem 1: The identity element of a group is unique.
Proof: Suppose *e*, *f* are identities for a group G.
Then $e = e\,f$ (since *f* is an identity) $= f$ (since *e* is an identity).☺🖐

Theorem 2: Each element of a group has only one inverse.
Proof: Suppose *b*, *c* are both inverses of the element *a* in a group *G*.
Then $b = b1 = b\,(ac) = (ba)\,c = 1c = c$. ☺🖐

NOTE: It is the associative law that ensures that inverses are unique. Algebraic systems that are not associative can indeed have multiple inverses. For example, the following system is closed and has an identity. Moreover every element has an inverse. In fact some elements have *two* inverses. But it isn't a group because it's not associative.

	1	a	b
1	1	a	b
a	a	1	1
b	b	1	1

Note that (aa)b = 1b = b while a(ab) = a1 = a.

Theorem 3: (Cancellation Law):
If $ax = ay$ in a group then $x = y$.
Proof: Suppose $ax = ay$. Then $a^{-1}(ax) = a^{-1}(ay)$.
Hence $(a^{-1}a)x = (a^{-1}a)y$. Thus $1x = 1y$ and so $x = y$. ☺✋

Notice that all the group axioms (except the Closure Law) are needed to prove this.

Similarly one can prove that $xa = ya$ implies that $x = y$. Thus cancellation on the left, or on the right, is possible in a group. A consequence of the cancellation law is that:

Every element appears exactly once and only once in each row and column of the group table.

You should take careful note of the fact that you can't cancel on the left-hand end of one side of the equation and on the right-hand end of the other. In other words $ax = xb$ does <u>not</u> imply that $a = b$.

§4.4. Subgroups

Recall the definition of subgroup. If G is a group and H is a subset of G then H is a **subgroup** if

(1) $xy \in \mathrm{H}$ for all $x, y \in \mathrm{H}$
(2) $1 \in \mathrm{H}$
(3) $x^{-1} \in \mathrm{H}$ for all $x \in \mathrm{H}$.

A subgroup H is a group in its own right. Every group is a subgroup of itself. All other subgroups are called **proper subgroups**. The **trivial subgroup**, **1** = {1} is a subgroup of any group.

Notation: H ≤ G means ‘H is a subgroup of G’ and **H < G** means ‘H is a proper subgroup of G’, that is, H is a subgroup but is not G itself.

Example 1: $2\mathbb{Z}$ (the group of even integers) is a proper subgroup of $\mathbb{Z}$ (under +).

Theorem 4: For all $g \in G$, $\langle g \rangle$ is a subgroup of G.
Proof: For all $r, s \in \mathbb{Z}^+$, $g^r g^s = g^{r+s} \in \langle g \rangle$; $1 = g^0 \in \langle g \rangle$ and for all r, $(g^r)^{-1} = g^{-r} \in \langle g \rangle$. ☺🖐

§4.5. Powers

If g is an element of a group, we define positive integer powers of g inductively as follows:

$g^0 = 1$;
$g^{n+1} = g^n g$ for all $n \geq 0$.

We define negative powers by $g^{-n} = (g^{-1})^n$ for all negative integers $-n$.

Theorem 5: For all natural numbers m, n and all elements a, b in a group G:
(1) $a^m a^n = a^{m+n}$; (2) $(a^m)^n = a^{mn}$;
(3) $(b^{-1}ab)^n = b^{-1}a^nb$; (3) if G is abelian, $(ab)^n = a^nb^n$.

Proof: Although these seem obvious enough (and indeed they are obvious if m and n are positive, just by counting factors) the cases where one or both are negative require special attention.

(1) This is obvious if both m, n are positive or zero. Suppose m is positive and n is negative, say $n = -r$ where r is positive. If $m \geq r$ then on the left hand side there will be r cancelling pairs of aa^{-1} leaving $m - r = m + n$ factors. If $m < r$ there will be only m such pairs leaving $r - m$ factors of a^{-1}. The result is therefore $a^{-(r-m)} = a^{m-r} = a^{m+n}$.

We've thus proved the result for all n where $\mathrm{m} \geq 0$. If m is negative, say $m = -s$, then putting $b = a^{-1}$ the left hand side is $b^s b^{-n}$. By the earlier case this is $b^{s-n} = a^{n-s} = a^{m+n}$.

(2) Again this is obvious if m, n are both positive. The other cases are left as an exercise.

(3) If n is positive we simply count the number of factors on each side. Because the group is assumed to be abelian the factors may be rearranged so all the a's can be brought to the front. If n is zero, LHS = RHS = 1. If n is negative we put $b = a^{-1}$ and use the positive case.

(4) If n is positive, $(\mathrm{b}^{-1}\mathrm{ab})^n = \mathrm{b}^{-1}(\mathrm{bb}^{-1})\, \mathrm{a}(\mathrm{bb}^{-1}) \ldots (\mathrm{bb}^{-1})\mathrm{ab}$ (n factors) $= b^{-1}aa \ldots ab$

$$= b^{-1}a^n b.$$

If $n = 0$, LHS = RHS = 1. If n is negative we use the positive case on $b^{-1}cb$ where c is defined to be a^{-1}. ☺✋

Theorem 6: In a group G, $(ab)^{-1} = b^{-1}a^{-1}$.
Proof: $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)\, b = b^{-1}b = 1$ and similarly $(ab)(b^{-1}a^{-1}) = 1$. ☺🖐
Remember that the inverse of a product is the product of the inverses **in reverse order**.

The **cyclic subgroup** generated by an element g is the set of all powers of g (including 1 as g^0 and negative powers). It's denoted by $\langle \boldsymbol{g} \rangle$.

Definition: The **order** of an element g of a group is the smallest positive integer n such that $g^n = 1$ (if such an n exists). In additive notation this becomes $ng = 0$.
The identity element of any group is the only element of order 1.

The order of g is denoted by $|\boldsymbol{g}|$. If there's no such positive n, we say that g has **infinite order**. For example the order of i in the group of non-zero complex numbers under multiplication is 4 since $i^4 = 1$ while no lower positive power is equal to 1. The number 2 has infinite order.

Theorem 7: The order of an element $g \in G$ is at most $|G|$.
Proof: Let $n = |G|$. The elements $1, g, g^2, \ldots g^{n-1}, g^n$ can't be distinct (there are n+1 of them in a group of order n). Hence there must be some repetition: $g^r = g^s$ for some r, s with $0 \le r < s \le n$. Thus $g^{s-r} = 1$ and since $0 < r < s \le n$, we must have $|g| \le n$. ☺🖐

Later, we'll prove that, in fact, the order of an element of *G divides* the order of G.

Theorem 8: Groups of even order must contain an element of order 2.

Proof: Suppose |G| is even. Now the elements of G that differ from their inverses must come in pairs $\{x, x^{-1}\}$. Since |G| is even, the remaining elements, those for which $x = x^{-1}$, must also be even in number. But $x = x^{-1}$ is equivalent to $x^2 = 1$ and so these are the elements of order 2, together with the identity. Leaving out the identity, there must be an odd number of elements of order 2 and so the number of elements of order 2 must be at least 1. ☺✋

Theorem 9: If all of the elements of G (except 1) have order 2, then G must be abelian.

Proof: Let $x, y \in$ G. Then $(xy)^2 = 1$. But also $x^2y^2 = 1$ and so $xyxy = xxyy$.

Multiplying by x^{-1} on the left and by y^{-1} on the right of each side of this equation we conclude that $yx = xy$. Since this holds for all $x, y \in$ G it follows that G is abelian. ☺✋

§4.6. Cyclic Groups

A group G is **cyclic** if it can be generated by a single element, that is if G = $\langle\ g\ \rangle$ for some $g \in$ G.

Example 2: The set of *n*-th roots of unity:

$$\{z \in \mathbb{C} \mid z^n = 1\}$$

is a group under multiplication. It is a cyclic group because it can be generated by $e^{2\pi i/n}$. This is because every *n*-th root of 1 has the form $e^{2k\pi i/n} = (e^{2\pi i/n})^k$. In particular the group of 4-th roots of unity is

$$\{1, i, -1, -i\}$$

which is generated by i.

Example 3: The group of symmetries of a parallelogram is {I, R} where R is a 180° rotation about its centre. This group is a cyclic group generated by R.

Example 4: The group $(\mathbb{Z}, +)$ of integers under addition is cyclic because it can be generated by the integer 1. Remember that we're using additive notation here so instead of saying that every integer is an integer *power* of 1 (which is not the case), we should be saying that every integer is an integer *multiple* of 1 (which it is). Note that −1 also generates this group, but ±1 are the only generators.

Theorem 10: Cyclic groups are abelian.

Proof: Two typical elements in the cyclic group $\langle g \rangle$ are g^r and g^s. Now $g^r g^s = g^{r+s} = g^s g^r$. So every pair of elements commute and hence the cyclic group is abelian. ☺✋

Theorem 11: A finite group of order n is cyclic if and only if it contains an element of order n.

Proof: Suppose that $|G| = n$ and G has an element g of order n. Then $\langle g\rangle$, the cyclic subgroup generated by g, has order n (that is there are n distinct powers of g). Thus there are no other elements in G. They're all powers of g and so g is a generator for G and hence G is cyclic.

Conversely suppose that G is a cyclic group of order n. Then, if g is a generator, g must have order n. ☺🖐

Example 5: The group given by the following group table isn't cyclic since it has no element of order 4.

	A	**B**	**C**	**D**
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

§4.7. Cosets and Lagrange's Theorem

Let $H \le G$. Define a relation $\equiv$ by defining $x \equiv y$ if $x = yh$ for some $h \in H$.

Theorem 12: $\equiv$ is an equivalence relation.

Proof: *Reflexive* Let $a \in G$. Then $a = a1$.

Since $1 \in H$, $a \equiv a$.

Symmetric Suppose $a \equiv b$. Then $a = bh$ for some $h \in H$. Hence $ah^{-1} = b$.

Since $h^{-1} \in \mathrm{H}$, $b \equiv a$.
Transitive Suppose $a \equiv b$ and $b \equiv c$. Then $a = bh$ for some $h \in \mathrm{H}$ and $b = ck$ for some $k \in \mathrm{H}$.
Thus $a = (ck)h = c(kh)$. Since $kh \in \mathrm{H}$, $a \equiv c$. ☺🖐
NOTE: Each of the three properties of an equivalence relation comes from one of the three closure properties of a subgroup.

The equivalence classes under $\equiv$ are called the **right cosets of H in G**.
Notation: $\boldsymbol{g}$**H** denotes the right coset containing $\boldsymbol{g}$.
NOTE: Left cosets are defined similarly.
Note also that some books define left and right cosets in the opposite manner.

Example 6: G = $\mathbb{R}^{\#}$, H = {±1} under the operation of multiplication. The cosets are all of the form $\{\pm x\}$.

Example 7: G = $\mathbb{C}^{\#}$, the group of non-zero complex numbers under multiplication, and let H be the subgroup of all complex numbers with modulus 1. The right (or left) cosets of H in G are all the circles with centre 0.

Example 8: Suppose G = $\mathbf{S}_3$ and H = {I, (12)}.
The left and right cosets of H in G are:
HI = {I.I, (12)I} = {I, (12)}   IH = {I.I, I(12)} = {I, (12)}
H(13) = {I(13), (12)(13)} = {(13), (123)}
(13)H = {(13)I, (13)(12)} = {(13), (132)}
H(23) = {I(23), (12)(23)} = {(23), (132)}

(23)H = {(23)I, (23)(12)} = {(23), (123)}.
Notice that in this case, left and right cosets are different.

Example 9: G = $\mathbf{S}_3$,   H = {I, (123), (132)}. Here the left and right cosets give the same decomposition of G. These two (left/right) cosets are H itself and {(12), (13), (23)}.

Theorem 13:
(1) The subgroup H is itself one of the cosets of H in G.
(2) Two elements a, b belong to the same right coset (of H in G) if and only if $b^{-1}a \in$ H.
Proof:
(1) H = 1H.
(2) a, b belong to the same right coset if and only if $a \equiv b$, that is, if and only if $a = bh$ for some $h \in$ H, that is, if and only if $b^{-1}a = h$ for some $h \in$ H, or more simply,
$b^{-1}a \in$ H. ☺🖐

Equivalence classes in general can be of different sizes, but cosets of a given subgroup must have the same size.

Theorem 14: If H ≤ G, every coset of H in G has |H| elements.
Proof: There's a natural 1-1 correspondence between any coset aH and H itself viz. $ah \leftrightarrow h$. Hence the number of elements in each is the same. ☺🖐

Theorem 15: (LAGRANGE) The order of a subgroup of a finite group divides the order of the group.
Proof: Suppose there are m cosets of H in G. Since G is the disjoint union of them and each coset has $|H|$ elements, it follows that $|G| = m. |H|$ and so $|H|$ divides $|G|$. ☺✋

This is a very powerful result. It shows that the number of elements in a group greatly affects its structure.

Example 10: If $|G| = 14$, the only possible orders for a subgroup are 1, 2, 7 and 14.

Theorem 16: Groups of prime order are cyclic.
Proof: Suppose $|G| = p$ where p is prime.
Since $p \geq 2$ we may choose $g \in G$ such that $g \neq 1$.
Let $H = \langle g \rangle$.
Let $|H| = n$. Now $n|p$ and $n > 1$ so $n = p$. Hence $G = H$ and so G is cyclic. ☺✋

Since the order of an element is the order of the cyclic subgroup it generates, we have:

Theorem 17: The order of an element of a finite group divides the order of the group. ☺

Lagrange's Theorem is a powerful one. But its converse does not hold in general. Just because a number divides the group order doesn't mean there has to be a subgroup of that order. For example $\mathbf{A}_4$ has order 12, but no subgroup of order 6.

However if a prime power divides |G| there is at least one subgroup of that order. This is part of what is known as the Sylow Theorems. Here we prove the special case where the divisor is just a prime.

Theorem 18 (CAUCHY): If p is a prime divisor of |G| there is an element of G of order p.

Proof: Consider all equations of the form $g_1g_2 \dots g_p = 1$ where the $g_i \in$ G.

There are $|G|^{p-1}$ such equations, because the first $p - 1$ factors can be chosen arbitrarily and the last one has to be the inverse of their product.

Now if $g_1g_2 \dots g_p = 1$ is one of these equations then so is $g_2g_3 \dots g_pg_1 = 1$. We just multiply both sides on the left by g_1^{-1} and on the right by g_1. In fact we can bring any number of the factors from the left-hand side and bring them to the right. Any cyclic rearrangement of the factors will also give one of these equations.

You might think that the set of these equations can be decomposed into sets of size p in this way. But what if every factor is equal? For example the cyclic rearrangements of the equation 1.1. … 1 = 1 will give just one equation not p distinct equations.

While ever two of the factors are different the cyclic rearrangements will be distinct. Notice that it is important for p to be prime for this to work because if p =

6 then *ababab* = 1 would only have two distinct cyclic arrangements.

Now if G has no element of order *p* then 1.1 … 1 = 1 is the only equation that is by itself. All the others can be decomposed into sets of size *p*.

But this would mean that the number of equations of the above form is congruent to 1 modulo *p*, which is impossible for $|G|^{p-1}$, since *p* divides |G|. ☺✋

§4.8. Dihedral Groups

The family of cyclic groups contains those groups with the simplest possible group structure. A closely related family is the family of dihedral groups.

The **dihedral group** of order 2*n* is the group:

$$\mathbf{D}_{2n} = \langle A, B \mid A^n = 1, B^2 = 1, BA = A^{-1}B \rangle.$$

Dihedral groups occur naturally in many different guises. $\mathbf{D}_{2n}$ is, for example, the group structure of the symmetry group of a regular *n*-sided polygon. The symmetry operations consist of the rotation R through $2\pi/n$, and its powers plus the 180° rotations about the *n* axes of symmetry. But if Q denotes any one of these, the others can be expressed in the form R^kQR^{-k}.

If Q denotes the 180° degree rotation about any one axis of symmetry (say a vertical axis), any other 180° symmetry operation can be achieved by rotating that axis until it becomes vertical (by some power R^k), carrying out Q about the vertical axis, and then rotating the axis back to its original position (by R^{-k}). Hence it can be expressed as R^kQR^{-k}.

Now $R^n = 1$ (n successive rotations through $2\pi/n$);

$Q^2 = 1$ (two successive 180° rotations) and

$QRQ = R^{-1}$ (a negative or clockwise rotation can be achieved by rotating about the vertical axis first, then rotating in the positive direction and finally rotating back in the vertical axis – try it!) and so $RQ = QR^{-1}$.

So this symmetry group is:

$$\langle R, Q \mid R^n = Q^2 = 1, RQ = QR^{-1}\rangle,$$

that is, it is the dihedral group of order $2n$. In particular D_8 is the symmetry group of a square.

§4.9. Dihedral Arithmetic

The dihedral group

$$\mathbf{D}_{2n} = \langle A, B \mid A^n = B^2 = 1, BA = A^{-1}B\rangle$$

has three relations. Let's examine their implications.

$\mathbf{A^n = 1}$**:** This means that any expression involving A's and B's need not have any string of successive A's longer than $n - 1$, because any block of n successive A's is A^n which, because it's equal to the identity, can be removed.

For example in

$$\mathbf{D}_4 = \langle A, B \mid A^4 = B^2 = 1, BA = A^{-1}B\rangle,$$

an element such as A^2BA^7BA can be simplified to A^2BA^3BA by removing an A^4 from the middle.

$B^2 = 1$: This means that it is never necessary to have two successive B's. For example an expression such as $AB^3A^2B^8A^3B$ can be simplified to ABA^5B by using $B^2 = 1$. In $\mathbf{D}_8$ this can be further reduced to ABAB by use of the relation $A^4 = 1$.

Another consequence of $B^2 = 1$ is the fact that $B = B^{-1}$ (just multiply both sides on the left by B^{-1}). This means that there's never any need to have B^{-1} in any expression.

$BA = A^{-1}B$: It is this third relation that makes dihedral groups non-abelian (except for the trivial cases of $\mathbf{D}_4$ and $\mathbf{D}_2$ where $A^{-1} = A$). Expressing this relation in words, we can say that every time a B passes across an A it inverts it, that is, converts it to A^{-1}.

Consequently if we have an expression involving a mixture of A's and B's we can move all the A's up to the front and all the B's down to the back just as we would if the commutative law was in force. The difference is that the A's get inverted every time a B crosses over. This is the dihedral 'twist'.

Example 13: In a dihedral group the expression ABA^3BA^2BAB can be written as:

$$AA^{-3}BBA^2BAB = A^{-2}BBA^2BAB$$
$$= A^{-2}A^2BAB = BAB = A^{-1}B^2 = A^{-1}.$$

Theorem 21: The elements of $\mathbf{D}_{2n}$ are:

1	A	A^2	A^3	…	A^{n-1}
B	AB	A^2B	A^3B	…	$A^{n-1}B$

Proof: Because of the relation $BA = A^{-1}B$ we can express every element in the form A^iB^j. But because $A^n = 1$ and $B^2 = 1$, we may assume that $i = 0, 1, 2, \ldots , n - 1$ and $j = 0$ or 1. ☺✋

Notice that the first row consists of the cyclic subgroup, H, generated by A, and the second row is the left coset HB.

Theorem 22: In the dihedral group

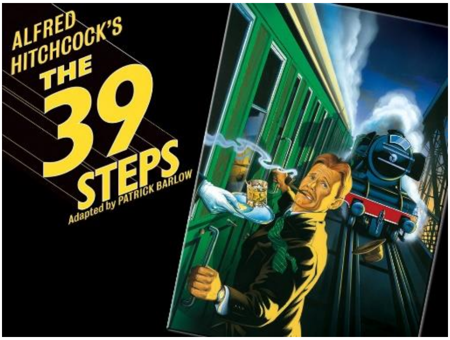
$$\mathbf{D}_{2n} = \langle A, B \mid A^n = B^2 = 1, BA = A^{-1}B\rangle$$

the elements of the form A^kB all have order 2.

Proof: $(A^kB)^2 = A^kBA^kB = A^kA^{-k}BB = BB = 1$. ☺✋

§4.10. Groups of Order 2*p*

We now classify groups of order 2*p* (where p is prime). We show that all such groups are either cyclic or dihedral. Since the proof is rather lengthy we've broken the argument up into 39 steps (with apologies to John Buchan and Alfred Hitchcock).



Also, to make it easier to see which assumptions are in force at any given time, we've used a similar indenting convention to that advocated when writing computer programs.

Theorem 23: If $|G| = 2p$ for some prime p then G is cyclic or dihedral.

Proof:

(1) Suppose that $|G| = 2p$ where p is prime and suppose that G is not cyclic.

(2) Suppose that p is odd.

(3) By Lagrange's theorem the order of every element is 1, 2, p or $2p$.

(4) Suppose G contains an element of order $2p$.

(4) Hence G is cyclic, a contradiction!

(5) Since $|G|$ is even, G must contain an element, b, of order 2.

(6) Suppose that *all* the elements of G, except 1, have order 2.

(7) Then G is abelian.

(8) Choose $a, b \in G$ of order 2 with $a \neq b$.

(9) Hence $H = \{1, a, b, ab\}$ is a subgroup of G of order 4.

(10) Since 4 doesn't divide $2p$ we get a contradiction.

(11) So G must have an element, a, of order p.

(12) Let $H = \langle a \rangle$.

(13) Since 2 doesn't divide p, $b \notin H$.

(14) The right cosets of H in G must be H and bH.

(15) Similarly the left cosets are H and Hb.

(16) Since bH and Hb consist of all the elements outside H, they must be equal, ie. Hb = bH.

(17) Now $ba \in b$H so $ba \in$ Hb.

(18) Hence $ba = a^r b$ for some integer

$r = 0, 1, \ldots , p - 1$.

(19) Hence $a = b^2 a = ba^r b$

(20) $= baa \ldots ab$

(where there are r factors of a).

(21) $= a^r a^r \ldots a^r\, b^2$

(where there are r factors of a^r).

(22) $= a^{r^2}$ (since $b^2 = 1$).

(23) So $a^{r^2} = a$, that is $a^{r^2 - 1} = a$.

(24) Hence p divides $r^2 - 1$.

(25) Since p is prime and $r^2 - 1 = (r - 1)(r + 1)$,

$p \mid r - 1$ or $p \mid r + 1$.

(26) Thus $ba = ab$ or $a^{-1}b$.

(27) Suppose $ba = ab$.

(28) Since G is not cyclic the order of ab must be 1, 2 or p.

(29) Suppose $ab = 1$.

(30) Then $a = b^{-1} = b$, a contradiction.

(31) Suppose ab has order 2.

(32) Then $1 = (ab)^2 = a^2 b^2 = a^2$, a contradiction.

(33) Suppose ab has order p.

(34) Then $1 = (ab)^p = a^p b^p = b^p$, a contradiction.

(35) So $b^{-1}ab = a^{-1}$ and so

$$G = \langle a, b \mid a^p = b^2 = 1, ba = a^{-1}b \rangle = \mathbf{D}_{2p}.$$

(36) Suppose that $p = 2$.

(37) Then every element of G (except 1) has order 2.

(38) Hence G is abelian.

(39) Hence $G = \langle a, b \mid a^2 = b^2 = 1, ba = ab \rangle$

$$= \langle a, b \mid a^2 = b^2 = 1, ba = a^{-1}b \rangle = \mathbf{D}_4.$$ ☺✋

EXERCISES FOR CHAPTER 4

EXERCISE 1: Show that the set of all matrices over $\mathbb{Z}_2$ of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is a group of order 8 under matrix multiplication. Does it satisfy the commutative law?

EXERCISE 2: Find all the elements of order 4 in the above group.

EXERCISE 3: Show that the above group is a dihedral group of order 8, that is find an element A of order 4 and an element B of order 2 such that $\mathrm{BA} = \mathrm{A}^{-1}\mathrm{B}$.

EXERCISE 4: Find all the elements of order 4 in

$$\mathbb{Z}_2 \oplus \mathbb{Z}_8.$$

EXERCISE 5: How many elements of the group $\mathbb{C}^{\#}$ of non-zero complex numbers under multiplication have order 7? How many have order 8?

EXERCISE 6: The following is a partially completed group table for a group. Complete it.

	a	***b***	***c***	***d***
a		*a*		
b			*c*	
c			*a*	
d		*d*		

Which element is the identity?

EXERCISE 7: Find all 4 possible group tables for a group G = {1, *a*, *b*, *c*} of order 4. Show that three of them are isomorphic (meaning that any one of them can be transformed to any other by a suitable relabelling) and the fourth is fundamentally different (eg look at the number of elements of order 2).

EXERCISE 8: If G is the group whose table is given below, show that H = {1, *c*, *d*} and K = {1, *b*} are both subgroups of G. Find all the left and right cosets of each subgroup.

	1	***a***	***b***	***c***	***d***	***e***
1	1	*a*	*b*	*c*	*d*	*e*
a	*a*	1	*c*	*b*	*e*	*d*
b	*b*	*d*	1	*e*	*a*	*c*
c	*c*	*e*	*a*	*d*	1	*b*
d	*d*	*b*	*e*	1	*c*	*a*
e	*e*	*c*	*d*	*a*	*b*	1

EXERCISE 9:
If G = $\langle a, b \mid a^4 = b^3 = 1, ab = ba\rangle$ and H is the cyclic subgroup generated by *b*, find the right and left cosets of H in G.

EXERCISE 10: In the dihedral group $\mathbf{D}_{10} = \langle a, b \mid a^5 = b^2 = 1, ba = a^{-1}b\rangle$, simplify

(i) $a^7b^3a^{-2}baba^3ba^2a^7b^2a$;

(ii) $a^{13}b^5a^2b^{-7}a^2ba$.

EXERCISE 11:

If |G| = 68, find the order of H given the following clues:

(a) H ≤ G and |H| < 32.

(b) H is non-cyclic.

EXERCISE 12:

Find |H| given the following clues.

(a) H is a subgroup of some group of order 100.

(b) H contains no element of order 2.

(c) H is not cyclic.

EXERCISE 13: Find |H| given the following clues:

(a) H is a subgroup of some group G of order 168.

(b) H is a subgroup of another group K of order 112.

(c) H is not cyclic or dihedral.

(d) H contains an element of order 7.

(e) H has more than two left cosets in K.

EXERCISE 14: If G is a group, the *centre* (*zentrum* in German) of G is defined to be $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. In other words, it's the set of all elements that commute with everything.

(a) Prove that Z(G) is a subgroup of G.

(b) Find $Z(\mathbf{D}_{2n})$. [**HINT:** You'll need to consider odd and even values of n separately.]

EXERCISE 15: Find the numbers of elements of each order in the following two groups whose group tables are given:

(a)

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*	1	*c*	*b*
b	*b*	*c*	1	*a*
c	*c*	*b*	*a*	1

(b)

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*	*b*	*c*	1
b	*b*	*c*	1	*a*
c	*c*	1	*a*	*b*

EXERCISE 16: Find the orders of the elements in the cyclic group of order 6.

EXERCISE 17: Find the orders of the elements of the following three groups: $G = \mathbb{Z}_{8,}$ $H = \mathbb{Z}^{\#}{}_{20}$, $K = \mathbf{D}_8$
Show that no two of these groups are isomorphic.

EXERCISE 18: Which of the above three groups of order 8 is cyclic? Which are abelian?

EXERCISE 19: Find $\langle 9\rangle$, the cyclic group generated by 9, in the group $\mathbb{Z}_{100}{}^{\#}$. This consists of all the integers from 1 to 99 that have no factors in common with 100. The operation is multiplication modulo 100. Also determine the order of 9 in this group.

EXERCISE 20: Find the order of the following elements in the group $\mathbb{Z}_{100}$ (consisting of all the integers from 0 to 99 under addition modulo 100): 2, 9, 6, 15.

EXERCISE 21:
Find the left and right cosets of {1, B} in the dihedral group

$$\mathbf{D}_{12} = \langle \mathrm{A}, \mathrm{B} \mid \mathrm{A}^6 = \mathrm{B}^2 = 1, \mathrm{BA} = \mathrm{A}^{-1}\mathrm{B}\rangle.$$

EXERCISE 22: G is the group whose table is given below. Show that H = {1, *a*, *d*, *f*} and K = {1, *d*} are both subgroups of G. Find all the left and right cosets of each subgroup.

	1	***a***	***b***	***c***	***d***	***e***	***f***	***g***
1	1	*a*	*b*	*c*	*d*	*e*	*f*	*g*
a	*a*	*d*	*e*	*g*	*f*	*c*	1	*b*
b	*b*	*g*	*d*	1	*c*	*a*	*e*	*f*
c	*c*	*e*	1	*d*	*b*	*f*	*g*	*a*
d	*d*	*f*	*c*	*b*	1	*g*	*a*	*e*
e	*e*	*b*	*f*	*a*	*g*	*d*	*c*	1
f	*f*	1	*g*	*e*	*a*	*b*	*d*	*c*
g	*g*	*c*	*a*	*f*	*e*	1	*b*	*d*

EXERCISE 23: Find the order of H given the following clues:

(a) H is a proper subgroup of a group of order 52;

(b) H is non-abelian.

EXERCISE 24: Find the order of H given the following clues:

(a) H is a subgroup of some group G of order 100.

(b) H is a subgroup of another group K of order 40.

(c) H is not cyclic or dihedral.

EXERCISE 25: Find the order of H given the following clues:

(a) H is a subgroup of some group G of order 20.

(b) H is non-abelian.

(c) G contains an element g of order 2 and an element h of order 5.

(d) H contains h but not g.

EXERCISE 26: Complete the following group table and find the orders of the elements.

	1	***a***	***b***	***c***	***d***	***e***
1						
a		1	*c*	*b*		*d*
b			*d*	*a*	1	
c		*d*	*e*		*a*	*b*
d		*c*		*e*	*b*	*a*
e		*b*	*a*		*c*	1

EXERCISE 27: Find the numbers of elements of each order in the cyclic group of order 12.

EXERCISE 28: Find the order of the following elements in the group $\mathbb{Z}_{17}^{\#}$ (consisting of all the integers from 1 to 16 under multiplication modulo 17): 2, 6, 9, 16

EXERCISE 29: Find the orders of the elements of the following three groups:

G = the group $\{\pm 1, \pm i, \pm(1 + i)/\sqrt{2}, \pm(1 - i)/\sqrt{2}\}$ under multiplication;

H = the group of symmetries of a rectangular box;

K = the group of order 8 whose group table is:

	1	**–1**	**i**	**–i**	**j**	**–j**	**k**	**–k**
1	1	–1	i	–i	j	–j	k	–k
–1	–1	1	–i	i	–j	j	–k	k
i	i	–i	–1	1	k	–k	–j	j
–i	–i	i	1	–1	–k	k	j	–j
j	j	–j	–k	k	–1	1	i	–i
–j	–j	j	k	–k	1	–1	–i	i
k	k	–k	j	–j	–i	i	–1	1
–k	–k	k	–j	j	i	–i	1	–1

Show that no two of these groups are isomorphic.

SOLUTIONS FOR CHAPTER 4

EXERCISE 1:

There are $2^3 = 8$ matrices of this form, over $\mathbb{Z}_2$.

$$\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a' & b'\\0 & 1 & c'\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+a' & b+b'+ac'\\0 & 1 & c+c'\\0 & 0 & 1\end{pmatrix}$$ so the set is closed under matrix multiplication. The associative law holds, as it always does with matrix multiplication. The identity matrix has this form.

Finally the inverse of $\begin{pmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{pmatrix}$ is $\begin{pmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{pmatrix}$ which has the required form.

Since $\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$ while $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$ the group doesn't satisfy the commutative law.

EXERCISE 2: $\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$.

EXERCISE 3: Take A = $\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$ and B = $\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$.

EXERCISE 4:

The elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ are vectors of the form (x, y) where $x \in \mathbb{Z}_2$ and $y \in \mathbb{Z}_8$.

If $4(x, y) = (0, 0)$ then $4x = 0 \bmod 2$ and $4y = 0 \bmod 8$.

This places no restriction on x: $x = 0$ or 1 but $4y = 0$ means $y = 0, 2, 4$ or 6.

But $(0, 0)$ is the identity and $(1, 0)$, $(0, 4)$ and $(1, 4)$ have order 2.

So the elements of order 4 are thus $(0, 2)$, $(0, 6)$, $(1, 2)$ and $(1, 6)$.

EXERCISE 5:

There are 6 elements of order 7, namely $e^{\pi i/7}, \ldots, e^{6\pi i/7}$ and 4 elements of order 8, namely $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$.

EXERCISE 6:

Since $bc = c$ the element b is the identity.

	a	***b***	***c***	***d***
a		*a*		
b	*a*	*b*	*c*	*d*
c		*c*	*a*	
d		*d*		

Now dc cannot be any of a, c or d because that would result in a repetition in either the row or the column corresponding to d. So dc must be b. In a similar way we can complete the table:

	a	**b**	**c**	**d**
a	*b*	*a*	*d*	*c*
b	*a*	*b*	*c*	*d*
c	*d*	*c*	*a*	*b*
d	*c*	*d*	*b*	*a*

EXERCISE 7: Filling out the entries for the identity we get:

	1	***a***	***b***	***c***
1	*1*	*a*	*b*	*c*
a	*a*			
b	*b*			
c	*c*			

Now *ab* = 1 or *c*.

Case I: *ab* = 1: We can now complete the group table:

	1	***a***	***b***	***c***
1	*1*	*a*	*b*	*c*
a	*a*	*c*	1	*b*
b	*b*	1	*c*	*a*
c	*c*	*b*	*a*	1

Case II: *ab* = *c*: The group table is thus:

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*		*c*	
b	*b*			
c	*c*			

Now *bc* = 1 or *a*.

Case IIA: *bc* = 1: The group table is thus:

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*		*c*	
b	*b*	*c*	*a*	1
c	*c*		1	

If $a^2 = b$ then $ac = 1$, a contradiction. Hence $a^2 = 1$ and so the group table is:

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*	1	*c*	*b*
b	*b*	*c*	*a*	1
c	*c*	*b*	1	*a*

Case IIB: *bc* = *a*: The group table is:

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*		*c*	
b	*b*	*c*	1	*a*
c	*c*		*a*	

This can be completed in two possible ways:

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*	1	*c*	*b*
b	*b*	*c*	1	*a*
c	*c*	*b*	*a*	1

	1	***a***	***b***	***c***
1	1	*a*	*b*	*c*
a	*a*	*b*	*c*	1
b	*b*	*c*	1	*a*
c	*c*	1	*a*	*b*

Of these four possibilities the ones that arise in Cases I and IIA are isomorphic to the second possibility under Case IIB. They therefore represent the same group with different notation. This group has just one element of order 2. The remaining possibility (the first under Case IIB) is essentially different in that it has 3 elements of order 2. So there are two groups of order 4. One is C_4 (cyclic) and the other is $\mathbf{D}_4 = V_4$ (dihedral).

EXERCISE 8:

The fact that H and K are subgroups of G can be most easily seen by extracting their group tables from the main table:

H	**1**	***c***	***d***
1	1	*c*	*d*
c	*c*	*d*	1
d	*d*	1	*c*

K	**1**	***b***
1	1	*b*
b	*b*	1

Since every entry in each table belongs to the subset in each case each subset is closed under multiplication. Clearly each contains the identity and, since 1 appears in each row and column, every element has an inverse within the respective subset.

The left cosets of H in G are:

H = {1, *c*, *d*} and
*a*H = (*a*1, *ac*, *ad*} = {*a*, *b*, *e*}.
The right cosets of H in G are:
H = {1, *c*, *d*} and
H*a* = {1*a*, *ca*, *da*} = {*a*, *e*, *b*}.

NOTE that in this example the left and right cosets are the same, even though the group is non-abelian.

The left cosets of K in G are:
K = {1, *b*}, *a*K = {*a*1, *ab*} = {*a*, *c*} and
*d*K = {*d*1, *db*} = {*d*, *e*}.

NOTE that we didn't waste our time with *b*K or *c*K because those elements were already included and we would have simply repeated the first two cosets.

For example *b*K = {*b*1, *bb*} = {*b*, 1} = {1, *b*}. So always use as a representative for a new coset, an element that hasn't yet been included.
The right cosets of K in G are:
K = {1, *b*}, K*a* = {1*a*, *ba*} = {*a*, *d*} and
K*c* = {1*c*, *bc*} = {*c*, *e*}
NOTE that in this case the left cosets and the right cosets give two different subdivisions of the group.

EXERCISE 9:

The elements are: 1, a, a^2, a^3,
b, ab, a^2b, a^3b,
b^2, ab^2, a^2b^2, a^3b^2.

The left cosets are $H = \{1, b, b^2\}$
$aH = \{a, ab, ab^2\}$
$a^2H = \{a^2, a^2b, a^2b^2\}$
and $a^3H = \{a^3, a^3b, a^3b^2\}$

Of course, since this group is abelian, the left cosets are the same as the right cosets.

EXERCISE 10:

(i) Using $b^2 = 1$ this becomes $a^7ba^{-2}baba^3ba^2a^7a$, and combining powers of a we get
$a^7ba^{-2}baba^3ba^{10}$. Using $a^5 = 1$ we get $a^2ba^3baba^3b$. Now we need to make use of the relation $ba = a^{-1}b$. Moving a b past an a, inverts it. Moving the second last b to the back we get $a^2ba^3baa^{-3}b^2 = a^2ba^3ba^3$. Moving the next b down gives $a^2ba^3a^{-3}b = a^2bb = a^2$.

(ii) $a^{13}b^5a^2b^{-7}a^2ba = a^3ba^2ba^2ba = a^3ba^2ba^2a^{-1}b = a^3ba^2bab = a^3ba^2a^{-1} = a^3ba = a^2b$.

EXERCISE 11:

By Lagrange's Theorem |H| divides 68. Since |H| < 32, |H| = 1, 2, 4 or 17. Since H is not cyclic we can eliminate 1, 2 and 17. Hence |H| = 4.

EXERCISE 12:

By Lagrange's Theorem |H| divides 100. H being non-cyclic rules out 1 and the primes 2 and 5, leaving 4, 5, 10, 20, 25, 50 and 100. Now groups of even order must contain an element of order 2. Since H doesn’t, it must have odd order, leaving 25 as the only possibility.

EXERCISE 13:

By Lagrange’s Theorem, |H| divides both 168 = 8×3×7 and 112 = 16×7 and therefore must divide their greatest common divisor, which is 56. Since H contains an element of order 7,

|H| must be divisible by 7. This limits the possibilities to 7, 14, 28 and 56. Now since H is neither cyclic nor dihedral, |H| can’t be prime or twice a prime [groups of order p are cyclic; groups of order 2p are cyclic or dihedral]. This narrows down the possibilities to 28 and 56. Now if |H| was 56 there would be exactly 2 left cosets in K which has order 112. By clue (e) this isn’t so, and hence 56 is ruled out. Therefore |H| must be 28.

EXERCISE 14:

(a) This is easily verified. Note that $gx = xg$ implies that $xg^{-1} = g^{-1}x$.

(b) $\mathbf{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b\rangle$. If $a^r \in Z(G)$ then $a^r b = ba^r = a^{-r}b$, so $a^{2r} = 1$ which means that $n \mid 2r$.

If n is odd this means $n \mid r$ and so $a^r = 1$.

If n is even a^r is 1 or $a^{n/2}$.

Similarly one can check that no element of the form $a^r b$ commutes with a. So $Z(\mathbf{D}_{2n}) = \{1\}$ if n is odd and
$\{1, a^{n/2}\}$ if n is even.

EXERCISE 15:
(a) has the identity plus 3 elements of order 2.
(b) has the identity, one element of order 2 (viz. b) and 2 elements of order 4.
[The fact that they differ in their structure in this way means that they're non-isomorphic, or essentially different. These two tables reflect the only two possible group structures for a group of order 4.]

EXERCISE 16:
The cyclic group of order 6 has the form:

$$\{1, g, g^2, g^3, g^4, g^5\} \text{ where } g^6 = 1$$

Clearly g has order 6.
$(g^2)^2 = g^4$, $(g^2)^3 = g^6 = 1$ and so g^2 has order 3.
$(g^3)^2 = 1$ and so g^3 has order 2.
$(g^4)^2 = g^8 = g^2$, $(g^4)^3 = g^{12} = 1$ and so g^4 has order 3;
The powers of g^5 are g^5, $g^{10} = g^4$, $g^{15} = g^3$, $g^{20} = g^2$, $g^{25} = g$.
Finally $(g^5)^6 = g^{30} = 1$ and so g^5 has order 6.
The cyclic group of order 6 thus has:
1 element of order 1;
1 element of order 2;
2 elements of order 3;
2 elements of order 6.

NOTE: orders 4 and 5 are missed out. Can you guess why?

EXERCISE 17:

G: 1, 3, 5, 7 have order 8
2, 6 have order 4
4 has order 2
0 has order 1

H: 3, 7, 13 and 17 have order 4
9, 11 and 19 have order 2
1 has order 1

K: The dihedral group $\langle A, B \mid A^4 = B^2 = 1, BA = A^{-1}B \rangle$ has elements: 1, A, A^2, A^3, B, AB, A^2B, A^3B. Of these: A, A^3 have order 4. A^2, B, AB, A^2B, A^3B have order 2 and 1 has order 1.

Listing the numbers of the elements of orders 1, 2, 4 and 8 respectively, as vectors we have:
G: (1, 1, 2, 4); H: (1, 3, 4, 0); K: (1, 5, 2, 0).
The differences show that these three groups are mutually non-isomorphic. There are in fact 5 distinct groups of order 8, the above three plus two others.

EXERCISE 18:

G is cyclic (and hence abelian) because it contains an element of order 8; H is abelian but not cyclic; K is non-abelian (and hence not cyclic).
NOTE: There's always only one cyclic group of any given order. In other words, all cyclic groups of order n

are isomorphic. A representative example of the cyclic group of order n is $\mathbb{Z}_n$ **=** $\{0, 1, ... n - 1\}$, of integers modulo n under addition.

EXERCISE 19:

The powers of 9 are:

$9^1 = 9$, $9^2 = 81$, $9^3 = 729 = 29$ mod 100 and so on. Rather than accumulate the higher and higher powers we can simply multiply by 9 at each stage to get the next:

$9^4 = 9 \times 29 = 261 = 61$

Then comes 49, 41, 69, 21, 89 and finally 1.

So $\langle 9 \rangle = \{1, 9, 81, 29, 61, 49, 41, 69, 21, 89\}$. There are 10 elements in this cyclic subgroup and so 9 has order 10 under multiplication modulo 100.

EXERCISE 20:

2: Remember that the operation is addition, so we need to keep adding the generator to itself, that is, taking higher and higher multiples, not powers. We want the smallest positive integer n such that $2n = 0$ mod 100, or in other words, such that $2n$ is a multiple of 100. The answer is clearly 50.

9: We want 100 to divide $9n$. Since 100 has no factors in common with 9, we'd need n itself to be a multiple of 100. The smallest positive such n is thus 100. So 9 has order 100 in this group.

6: We need $6n$ to be a multiple of 100. Since 2 divides both 6 and 100 we need 50 to divide $3n$. But since 50 has no factor in common with 3, we'd need 50 to divide *n*. So 6 has order 50.

15: 15*n* = 0 mod 100 means that 3*n* = 0 mod 20. Since 3 is coprime with 20, we need *n* = 0 mod 20, so 15 has order 20.

EXERCISE 21:

Left cosets:

1 B	A AB	A^2 A^2B	A^3 A^3B	A^4 A^4B	A^5 A^5B

Right cosets

1 B	A BA	A^2 BA^2	A^3 BA^3	A^4 BA^4	A^5 BA^5

i.e.

1 B	A A^5B	A^2 A^4B	A^3 A^3B	A^4 A^2B	A^5 AB

EXERCISE 22:

H and K contain 1 and are closed under multiplication and inverse as these tables show:

H	**1**	***a***	***d***	***f***
1	1	*a*	*d*	*f*
a	*a*	*d*	*f*	1
d	*d*	*f*	1	*a*
f	*f*	1	*a*	*d*

K	**1**	***d***
1	1	*a*
d	*d*	1

The left and right cosets of H are

1 a d f	*b c e g*

The left and right cosets of K are

1 d	*a f*	*b c*	*e g*

EXERCISE 23:

By (a) |H| = 1, 2, 4, 13, 26 and by (b) |H| = 26.

EXERCISE 24:

By (a) |H| divides 100. By (b) |H| divides 40. Hence |H| divides 20 and so |H| = 1, 2, 4, 10 or 20. By (c) |H| = 20.

EXERCISE 25:

By (a) |H| divides 20. By (c), (d) |H| is divisible by 5. Hence |H| = 5, 10 or 20.

By (b) |H| ≠ 5. By (d) |H| ≠ 20. Hence |H| = 10.

EXERCISE 26:

	1	***a***	***b***	***c***	***d***	***e***
1	1	*a*	*b*	*c*	*d*	*e*
a	*a*	1	*c*	*b*	*e*	*d*
b	*b*	*e*	*d*	*a*	1	*c*
c	*c*	*d*	*e*	1	*a*	*b*
d	*d*	*c*	1	*e*	*b*	*a*
e	*e*	*b*	*a*	*d*	*c*	1

EXERCISE 27:

If $G = \langle A \mid A^{12} = 1\rangle$ there are 4 elements of order 12 (A, A^5, A^7, A^{11}), 2 elements of order 6
(A^2, A^{10}), 2 elements of order 4 (A^3, A^9), 2 elements of order 3 (A^4, A^8), 1 element of order 2 (A^6) plus the identity of order 1.

EXERCISE 28:

The group has order 16 so the possible orders of the elements are powers of 2.
2: $2^2 = 4$, $2^4 = 16 = -1$, $2^8 = 1$ so 2 has order 8.
6: $6^2 = 2$ so 6 has order 8.
9: $9^2 = 13 = -4$, $9^2 = 16 = -1$, $9^4 = 1$ so 9 has order 4.
15: $16^2 = (-1)^2 = 1$ so 16 has order 2.

EXERCISE 29:

G is the group of 8th roots of 1.
The four elements $\pm(1 + i)/\sqrt{2}$, $\pm(1 - i)/\sqrt{2}$ have order 8, $\pm i$ have order 4, -1 has order 2 and 1 has order 1.
$H = \langle A, B, C \mid A^2 = B^2 = C^2 = 1, BA = AB, AC = CA, CB = BC\rangle$. It has 7 elements of order 2 plus the identity.
K: has 6 elements of order 4 ($\pm i$, $\pm j$, $\pm k$), only 1 element of order 2 (-1), plus the identity.
Since these three groups differ in the numbers of elements of each order no two of them can be isomorphic.